EXPECTED MULTI-UTILITY REPRESENTATIONS BY “SIMPLEX” WITH APPLICATIONS

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Expected multi-utility representations by "simplex" with applications

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Abstract

We give sufficient conditions to characterize the class of (possibly incomplete) preference relations over lotteries which can be represented by a Bauer simplex of (continuous) expected utility functions that preserve both indifferences and strict preferences. Our result is applied to a model of stochastic choice with the measurement of random expected utility functions and to a model of subjective expected utility with subjective states of the world.

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Keyword: Incomplete preferences, expected utility, random utility, random choice, subjective expected utility, states of the world.

1 Introduction

Starting with Aumann (1962), early research on representation of incomplete preference relations under risk explored sufficient conditions that allow one to extend a preference relation by a single expected utility function in a one-way representation. Concretely, given a preference relation (preorder) $\succsim$ over the set of lotteries on a prize space X, there exists a one-way representation by a von NeumannMorgenstern function u if for all lotteries p, q

$$p \succ (\sim) q \text{ implies } \int_X u \mathrm{d}p > (=) \int_X u \mathrm{d}q.$$

A major drawback of this approach is that one cannot recover the preference relation $\succsim$ from the utility function u. The problem of recovering the an incomplete preference relation gave rise to the literature on multi-utility representations which provide a set of utility functions that fully characterize a given preference relation. Following Dubra et al. (2004), there exists a multi-utility representation by a set U of von NeumannMorgenstern function if for all lotteries p, q

$$p \succsim q \text{ if and only if } \int_X u \mathrm{d}p \geq \int_X u \mathrm{d}q, \ \ \forall u \in U.$$

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By now, there is a large literature on multi-utility representations of preference relations. In this paper, we focus on multi-utility representations with some additional structure. We give sufficient conditions to characterize the class of (possibly incomplete) preference relations over lotteries which can be represented by a Bauer simplex of (continuous) expected utility functions that preserve both indifferences and strict preferences. The starting point of our paper is the work of Evren (2014) that provides axiom for multi-utility representations by compact sets. Considering simplices allows to decompose every utility function in a unique way as an integral over extremal utility functions.

In an other result, we provide a characterization of expected utility preferences over lotteries, by considering such a preference relation as a completion of an underlying, incomplete preference relation. It has been recently observed that a variety of interesting behavioural phenomena can be explained by two-stage choice procedures where in the first stage the individual identifies a collection of maximal alternatives, and then makes his final choice among these maximal alternatives according to a secondary criterion. Our findings are applied to two problems.

First, to a model of stochastic behaviour with the measurement of random expected utility functions. Historically, an important proportion of the research on choice and preference has placed the emphasis on deterministic representations, most notably for utility and uncertainty. A notorious challenge has been the question of adequate representation for variability in behaviour and in experiments, both across decision makers and within a person. Here we focus on probabilistic representations of utility, and consider deterministic representations as the special case where all probability mass is concentrated a single utility function. We investigate further the measurement process to deliver and discriminate between the supposed random utility function and the effectively measured random utility function of an individual.[1]

Second, to a model of subjective expected utility with subjective states of the world without the use of menu as in Dekel et al. (2001). In the representations, the individual acts as if he had coherent beliefs about what he knows with certainty about the alternatives. More precisely, subjective states are extremal sure ex ante preferences over lotteries. We show that this set of ex ante preferences, called the subjective state space, is essentially unique given the restriction that all ex ante preferences are expected-utility preferences. Because the subjective state space is identified, the way ex ante utilities are aggregated into an ex post ranking is also essentially unique given a set of state-dependent utilities.[2]

2 Main result

2.1 Setup and basic assumptions

Given a compact metric space X, which is the set of all certain prizes (degenerate lotteries), we denote by $C(X)$ the space of continuous real valued functions

[1]For instance, see Gul and Pesendorfer (2006), Blavatskyy (2008), Blavatskyy (2012), Dagsvik (2008), Gul et al. (2014) for recent studies of stochastic choice under risk on a quite different topic.

[2]Karni (2006) provides an axiomatic theory of decision making under uncertainty that *dispenses* with the state space. A different topic.

on X topologized by the sup-norm, and by $\mathcal{P}(X)$ the set of all (Borel) probability measures (lotteries) over X, endowed with the topology of weak convergence. We define a preference relation $\succsim$ as any preorder, that is a reflexive and transitive binary relation, on $\mathcal{P}(X)$. We denote by $\sim$ and $\succ$ the symmetric and asymmetric parts of $\succsim$, respectively, defined as usual: $p \sim q$ if and only if $p \succsim q$ and $q \succsim p$; and $p \succ q$ if and only if $p \succsim q$ and $q \not\succsim p$. We denote by $\mathbb{E}(p, u)$ the expectation of $u \in C(X)$ with respect to $p \in \mathcal{P}(X)$; that is,

$$\mathbb{E}(p, u) = \int_X p(x)u(x)dx.$$

A preference relation on $\mathcal{P}(X)$ is said to have a (continuous) expected multi-utility representation if there exists a set $U \subseteq C(X)$ such that, for any two lotteries $p, q \in \mathcal{P}(X)$,

$$p \succsim q \Leftrightarrow \mathbb{E}(p, u) \geq \mathbb{E}(q, u), \ \ \forall u \in U.$$

Given $\succsim$ on $\mathcal{P}(X)$ and $u \in C(X)$, u is said to be strictly $\succsim$-increasing if $p \succ q$ implies that $\mathbb{E}(p, u) > \mathbb{E}(q, u)$. A preference relation on $\mathcal{P}(X)$ is said to have a (continuous) expected Richter-Peleg multi-utility representation if it admits a (continuous) expected multi-utility representation by a family of strictly $\succsim$-increasing functions.

$U \subseteq C(X)$ is said to be closed (compact) if it is closed (compact) with respect to the topology of $C(X)$. Moreover, we denote by $co(U)$ the convex hull of U, by $\bar{co}(U_{p,q})$ the topological closure of $co(U)$, and by $\partial_e U$ the set of extreme points of U.[3] Finally we need the notion of simplex. Let $\mathcal{V}$ be a convex subset of a locally convex space E. Passing to $E \times \mathbb{R}$, $\mathbb{R}$ the scalar field, if necessary, we may suppose that $\mathcal{V}$ lies in a hyperplane of E which misses the origin. The set $\mathcal{V}$ is a simplex if the cone $C = \{\alpha v \mid \alpha \geq 0, v \in \mathcal{V}\}$ generated by $\mathcal{V}$ induces a lattice order in $C - C$.[4] If the set $\mathcal{V}$ is compact and it is a simplex, then $\mathcal{V}$ is said to be a Choquet simplex. Moreover, the set $\mathcal{V}$ is said to be a Bauer simplex if it is a Choquet simplex and if the set $\partial_e \mathcal{V}$ of its extreme point is closed.

The following assumptions are employed by Evren (2014) to characterize the class of preorders that admits an expected Richter-Peleg multi-utility representation by a compact set $U \subseteq C(X)$.

A1. Preorder. $\succsim$ is reflexive and transitive.

A2. Non-triviality. $p^* \succ q^*$ for some $p^*, q^* \in \mathcal{P}(X)$.

A3. Strong Independence. For any $p, q, r \in \mathcal{P}(X)$ and any $\alpha \in (0, 1)$,

$$p \succ (\sim) q \Leftrightarrow \alpha p + (1 - \alpha) r \succ (\sim) \alpha q + (1 - \alpha) r.$$

[3]Let E be a real vector space, the convex hull of a subset $F \subseteq E$ is the smallest convex subset of E that contains F; an extreme point of a convex set G is a point in G which does not lie in any open line segment joining two points of G.

[4]Note that in finite dimensional spaces a simplex coincides with the usual notion of simplex. Moreover, a simplex $\mathcal{V}$ may not, strictly speaking, generate a cone in E, e.g. if the origin is in the core of $\mathcal{V}$. More precisely, $\mathcal{V}$ is a simplex if $\mathcal{V}$ is affinely homeomorphic to a set with the stated properties. A cone C in a vector space V induces a binary relation $\geq$ on V, defined so that $x \geq y$ if and only if $x - y \in C$. A lattice order is a partial order in which every two elements have a unique least upper bound and a unique greatest lower bound.

A4. Open-continuity. *For any $q \in \mathcal{P}(X)$, the sets $\{p \mid p \succ q\}$ and $\{p \mid q \succ p\}$ are open subsets of $\mathcal{P}(X)$.*

A5. Symmetric-closedness. *For any $p, q \in \mathcal{P}(X)$, if p belongs to the closure of both $\{r \mid r \succ q\}$ and $\{r \mid q \succ r\}$, then $p \sim q$.*

Below, this is a slight rephrasing of the original (Evrens) theorem.

Theorem 1 (Evren (2014))**.** *A binary relation $\succsim$ on $\mathcal{P}(X)$ satisfies Axioms A1-A5 if and only if there exists a compact set $U \subseteq C(X)$ of strictly $\succsim$-increasing functions such that, for any two lotteries $p, q \in \mathcal{P}(X)$,*

$$p \succsim q \Leftrightarrow \mathbb{E}(p, u) \geq \mathbb{E}(q, u), \ \ \forall u \in U.$$

In the sequel, given a pair of lotteries p, q with $p \succ q$, we denote by $U_{p,q}$ a normalized representation for $\succsim$, that is, a set $U \subseteq C(X)$ of strictly $\succsim$-increasing functions that satisfies $\mathbb{E}(p, u) = 1$ and $\mathbb{E}(q, u) = 0$ for every $u \in U$. In the case of the previous Theorem, we have the following result

Proposition 1. *Let $\succsim$ be a binary relation on $\mathcal{P}(X)$ that satisfies Axioms A1-A5, and $U_{p,q} \subseteq C(X)$ be a normalized representation for $\succsim$. Then $\mathcal{V}_{p,q} \subseteq C(X)$ is another such representation if and only if $\bar{co}(U_{p,q}) = \bar{co}(\mathcal{V}_{p,q})$. Moreover, if $U_{p,q}, U_{p',q'} \subseteq C(X)$ are two normalized representations for $\succsim$ then there exists a homeomorphism h from $\bar{co}(U_{p,q})$ to $\bar{co}(U_{p',q'})$ such that the maps h and h^{-1} both preserve extreme points and faces.*[5]

The first part of this proposition is now classical for representation of incomplete preferences. In contrast, the second part specifies the link between different normalizations.

2.2 Representation by Bauer simplices

Consider the following axiom for a preorder on $\mathcal{P}(X)$

A6. Lattice property. *For any $p, q \in \mathcal{P}(X)$ there exist $r_1, s_1 \in \mathcal{P}(X)$ and $\alpha \in (0, 1)$ such that $r_1 \succsim s_1$, $\alpha r_1 + (1-\alpha)q \succsim \alpha s_1 + (1-\alpha)p$, and if $r_2, s_2 \in \mathcal{P}(X)$ and $\beta \in (0, 1)$ are such that $r_2 \succsim s_2$, $\beta r_1 + (1-\beta)q \succsim \beta s_1 + (1-\beta)p$, then*

$$\frac{\beta(1-\alpha)}{\alpha - 2\alpha\beta + \beta} r_2 + \frac{\alpha(1-\beta)}{\alpha - 2\alpha\beta + \beta} s_1 \succsim \frac{\beta(1-\alpha)}{\alpha - 2\alpha\beta + \beta} s_2 + \frac{\alpha(1-\beta)}{\alpha - 2\alpha\beta + \beta} r_1.$$

This axiom is a common lattice property for differences. For if we define $\mathcal{C} := \{\lambda(p-q) \mid \lambda \geq 0,\ p \succeq q\}$ and let $\mathcal{S}$ be the linear span of $\mathcal{P}(X) - \mathcal{P}(X)$ with the zero vector denoted 0. It is routine to show that $\mathcal{C}$ defines a preorder on $\mathcal{S}$. Equipped with such a preorder, denoted $\succsim$ by a slight abuse of notation, $\mathcal{S}$ becomes a preordered vector space. The lattice property implies that for all $x \in S$, there exists $y \in S$ such that $y \succsim x, 0$ and if $z \in S$ is such that $z \succsim x, 0$, then $z \succsim y$. It follows that $\mathcal{S}/\sim$ the partially ordered vector space induced by $\succsim$ is a lattice ordered vector space.[6] Note that it is a richness assumption on $\succsim$

[5] A face of a convex set K is a convex subset F of K (possibly empty) such that any line segment with endpoints in K whose interior meets F must be contained in F.

[6] See the proof of Proposition 2.

and not on the set $\mathcal{P}(X)$ as this axiom is trivially satisfied by a connected, hence complete, binary relation. It is difficult to say just what this means other than to restate it in words. Every two indifference classes of difference have a least upper bound and a greatest lower bound. For instance, this axiom is satisfied by the partial ordering defined by the first-order stochastic dominance relation. We recognize that this assumption restricts the degree of incompleteness of the preference relations under consideration. The next proposition states that this supplementary assumption are sufficient to obtain a representation by a Bauer simplex.

Proposition 2. *If a binary relation $\succsim$ on $\mathcal{P}(X)$ satisfies Axioms A1-A6, then there exists an expected Richter-Peleg multi-utility representation by a Bauer simplex $U \subseteq C(X)$.*

Remark 1. If X is a finite set, then U is a classical simplex, that is, U is the convex hull of an affinely independent subset $\mathcal{V}$ of $\mathbb{R}^X$. It then follows from the affine independence that $\mathcal{V}$ is the finite set of extreme points of U. Consequently, by the uniqueness of the representation, $\succsim$ admits a representation by a finite set of utilities.

2.3 Expected utility completion

An approach to the problem of modelling choice behaviour of a decision maker with incomplete preferences is to use a complete preorder (weak order) to represent the choices of the decision maker.[7] As a minimal consistency requirement, the complete binary relation that represents the choice behaviour must extend the underlying incomplete, psychological preference relation. We introduce the following assumptions to relate choices to preferences. Let $\succsim$ (as above) and $\succsim^o$ be two binary relations on $\mathcal{P}(X)$, The derived relations $\succ$, $\sim$, $\succ^o$, and $\sim^o$ are defined as usual.

A0. $\succsim^o$-Completeness. For any $p, q \in \mathcal{P}(X)$, $p \succsim^o q$ or $q \succsim^o p$.

A7. Consistency. For any $p, q \in \mathcal{P}(X)$, $p \succ q$ implies $p \succ^o q$, and $p \sim q$ implies $p \sim^o q$.

Proposition 3. *Let $\succsim$, $\succsim^o$ be two binary relations on $\mathcal{P}(X)$. Suppose that $\succsim$ satisfies A1-A6, $\succsim^o$ satisfies A0-A4, and that the pair $(\succsim, \succsim^o)$ jointly satisfy A7. Then there exist a compact set $U_{p^*,q^*} \subseteq C(X)$ (of strictly $\succsim$-increasing functions and unique up to homeomorphism), an unique regular Borel probability measure μ on $\mathcal{B}(U_{p,q})$ (the algebra of Borel sets of $U_{p,q}$), and an unique real-valued function $v \in C(X)$ that satisfies $\mathbb{E}(p^*, v) = 1$ and $\mathbb{E}(q^*, v) = 0$ such that for all $p, q \in \mathcal{P}(X)$*

(i) $p \succsim q \Leftrightarrow \mathbb{E}(p, u) \geq \mathbb{E}(q, u), \quad \forall u \in U_{p^*,q^*}$;

(ii) $p \succsim^o q \Leftrightarrow \mathbb{E}(p, v) \geq \mathbb{E}(q, v)$;

(iii) $\mathbb{E}(p, v) = \displaystyle\int_{U_{p^*,q^*}} \mathbb{E}(p, u)\mathrm{d}\mu(u)$.

[7] A complete preorder or weak order $\succsim$ on a set X satisfies $x \succsim y$ or $y \succsim x$, for all $x, y \in X$.

Note that the utility set and the probability measure that figure in the representation are unique as a pair, that is, the probability measure is unique given the utility set and the utility set is unique (up to a common positive affine transformation) given the probability measure. There are infinitely many distinct "utility-probability" pairs that represent the same preference relations in the sense of the previous proposition. Moreover, because probability distributions are distinct it is not evident to find some invariant pattern consistent with $\succsim$ and $\succsim^o$. However, we show in the next section that the probabilities that figure in the representation are not meaningless.

3 Economic applications

3.1 Measurement of random expected utility

Classical choice theory models choice behaviour as deterministic. Modelling choice behaviour as stochastic is useful and often necessary to represent the variability of decision maker's tastes. For instance, the choice behaviour of a group of individuals with "identical characteristics", each facing the same decision problem, presents the observer with a frequency distribution over outcomes. Characteristically, such data are interpreted as the outcome of independent random choice by a group of identical individuals. Even when repeated decisions of a single individual are observed, choice behaviour may exhibit variation and therefore suggest random choice. Foremost, random choice models take probability distributions as primitive concept.

In what follows, the choice objects are lotteries over a compact metric space. The decision maker faces binary choice problem between choice objects and he is characterized by two primitives binary relations $\succsim_d$ and $\succsim_s$ that reflect deterministic and stochastic patterns, respectively, of the choice behaviour. The interpretation is: $p \succsim_d q$ if and only if the decision maker prefers, deterministically, p to q; and $p \succsim_s q$ if and only if the decision maker prefers, stochastically (or in mean), p to q.

In the case of an outside observer which scrutinizes preferences, a consistent setting that describes our primitives could be: $p \succsim_d q$ holds if and only if the decision maker strictly prefers, systematically, p over q or the decision maker is indifferent, systematically, between p and q in repeated binary choice between p and q; $p \succsim_s q$ holds if and only if the decision maker strictly prefers, on average, p over q or the decision maker is indifferent, on average, between p and q in repeated binary choice between p and q. We may doubt of our capacity to perform endless choice, provided that their results exist. However, in ordinary practice, the existence of expensive and laborious experiments gives meaning to our definitions.

Suppose that $\succsim_d$ satisfies A1-A6, $\succsim_s$ satisfies A0-A4, and that the pair $(\succsim_d, \succsim_s)$ jointly satisfy A7. The interpretation is as follows: $\succsim_d$ is not complete because of the variability in preferences. $\succsim_s$ is complete so that we suppose explicitly that average preferences are made. The consistency axiom means that if the decision maker prefers, systematically, p to q, then he must, on average, prefer p over q. Apart from the lattice property the other axioms are quite standard, our random utility axiomatization relates random preferences to the simplest theory of choice under risk: expected utility theory. In this setting, the lattice property means that for every two lotteries there exists two lotteries r_1, s_1 and a real number $\alpha \in [0,1]$ with r_1 systematically better than s_1 and such that a deterministic comparison can be performed for the composed lotteries $\alpha r_1 + (1-\alpha)q$ and $\alpha s_1 + (1-\alpha)p$; moreover r_1, s_1 and α are "minimal". By Proposition 3, once the unit is fixed, every lottery is associated with a random variable of expected utility and the expectation of the random variable replaces the usual expected utility. This model account for the measurement of utility as a random variable from its elicitation. Observe that the result is a random variable about the measure of the utility and not about the existence of a random expected utility which is generated by, and generates, a random choice rule: a quite different topic.[8]

In view of Proposition 1, the significance of the uniqueness assertion is that we may change the unit of measurement without altering the sample space of the measurement process but we need to use a Radon-Nikodym type theorem to go from one representation to another. This is logical because we must express a new probability measure as an integral with respect to a reference measure.

3.2 Subjective expected utility without states of the world

A primitive concept of most of modern theories of decision-making under uncertainty is the state space, introduced by Savage (1954), the set of all states of the world with mutually exclusive and jointly exhaustive elements. Choice objects are acts, that is functions from states to consequences.

Let X be a compact metric space that represents a set of pure uncertainty-dependent consequences. Similarly to Anscombe and Aumann (1963) we augment the choice set considering the set of objective lotteries over X.[9] The uncertainty in the consequences is not resolved at the time of the choice, the decision maker is characterized by two primitives binary relations $\succsim_c$ and $\succsim_u$. $\succsim_c$ is interpreted as follows: $p \succsim_c q$ holds if and only if p is preferred to q whichever the uncertainty in the world. That is, regardless the resolution of the uncertainty, the decision maker prefers the choice object p over q. $\succsim_u$ is interpreted as follows: $p \succsim_u q$ holds if and only if p is preferred to q when the uncertainty in the world is taken into account. The decision-making process is decomposed into two sub-processes. The first is the evaluation of the subjectively "sure" preferences over uncertainty-dependent consequences. The second is the assessment of the choice objects with respect to the first process.

Suppose that $\succsim_c$ satisfies A1-A7, $\succsim_u$ satisfies A0-A4, and that the pair $(\succsim_c, \succsim_u)$ jointly satisfy A8. The interpretation is as follows: $\succsim_c$ is not complete because tastes depends of the resolution of the uncertainty. $\succsim_u$ is complete so that we suppose that the decision maker made a bet on the future resolution of the uncertainty. The consistency axiom means that if the decision maker prefers p to q whichever the uncertainty in the world, then he must prefer p over q when he made is final decision. In this setting, the lattice property means that for every two lotteries there exists two lotteries r_1, s_1 and a real number $\alpha \in [0,1]$ with r_1 surely better than s_1 and such that, regardless the uncertainty, comparison can be performed for the composed lotteries $\alpha r_1 + (1-\alpha)q$ and $\alpha s_1 + (1-\alpha)p$; moreover r_1, s_1 and α are "minimal". By a slight rephrasing

[8]See for instance Gul and Pesendorfer (2006).

[9]There is no contradiction in assuming existence of random devices.

of proposition 3, there exist a set S of jointly exhaustive elements, an utility function from $X \times S$ and an unique regular Borel probability measure μ on $\mathcal{B}(S)$ (the algebra of Borel sets of S) such that for all pure uncertainty-dependent consequences $x, y \in X$

(i) $x \succsim_c y \Leftrightarrow u(x,s) \geq u(y,s), \ \ \forall s \in S$;

(ii) $x \succsim^u y \Leftrightarrow \displaystyle\int_S u(x,s)\mathrm{d}\mu(s) \geq \int_S u(y,s)\mathrm{d}\mu(s)$.

It follows that the decision maker has a subjective state space and a state-dependent utility with respect to some normalization (we can assume without loss of generality that there exist $x^*, y^* \in X$ such that $x^* \succsim_c y^*$). The subjective expected utility representation separates risk attitudes, represented by the utility function, from beliefs, represented by the subjective probabilities. However, the uniqueness of the probabilities depends crucially on the convention of the normalization imposed to the state-dependent utility function. This premise is not implied by the axioms. There are infinitely many distinct utility-probability pairs that represent the same preference relations. Hence, it is not clear which of the infinitely many probability distributions consistent with the preference relation $\succsim^u$ actually represents the decision makers beliefs. However, this drawback exist in the classical models of Savage (1954) and Anscombe and Aumann (1963) and our contribution is about subjective state space.

4 Proofs

Proof of Proposition 1. Let $\succsim$ be a binary relation on $\mathcal{P}(X)$ that satisfies axioms A1-A5, and $U_{p,q} \subseteq C(X)$ be a normalized representation for $\succsim$. Evren (2014, Theorem 3) has already shown that $\mathcal{V}_{p,q} \subseteq C(X)$ is another such representation if and only if $\bar{co}(U_{p,q}) = \bar{co}(\mathcal{V}_{p,q})$. It remains to prove that, if $U_{p,q}, U_{p',q'} \subseteq C(X)$ are two normalized representations for $\succsim$ then there exists a homeomorphism h from $\bar{co}(U_{p,q})$ to $\bar{co}(U_{p',q'})$ such that the maps h and h^{-1} both preserve extreme points and faces. By assumption, $p' \succ q'$. For any $u \in \bar{co}(U_{p,q})$, we have $u(p') > u(q')$, and so $u(p') - u(q') > 0$. Thus the function h defined by

$$h(u) = \frac{u - u(q')}{u(p') - u(q')},$$

is strictly $\succsim$-increasing. This provides us with a function h from $\bar{co}(U_{p,q})$ to $\bar{co}(U_{p',q'})$, and we observe that h is continuous. Similarly, the rule

$$h'(v) = \frac{v - v(q)}{v(p) - v(q)},$$

defines a continuous function from $\bar{co}(U_{p',q'})$ to $\bar{co}(U_{p,q})$. Observe that $h' \circ h$ is the identity function on $\bar{co}(U_{p,q})$. Similarly, $h \circ h'$ is the identity function on $\bar{co}(U_{p',q'})$, and thus h and h' are inverse homeomorphisms. It is straightforward to show that whenever u in $\bar{co}(U_{p,q})$ is a positive convex combination of strictly $\succsim$-increasing functions u_1, u_2 in $\bar{co}(U_{p,q})$, then $h(u)$ is a positive convex combination of $h(u_1)$ and $h(u_2)$ in $\bar{co}(U_{p',q'})$.

Now let us consider any face F in $\bar{co}(U_{p',q'})$. Given a convex combination $u = \alpha_1 u_1 + \alpha_2 u_2$ with $u \in \bar{co}(U_{p,q})$ and $u_1, u_2 \in h^{-1}(F)$, $h(u)$ is a convex combination of the points $h(u_1), h(u_2)$ from F, whence $h(u) \in F$, and so $u \in h^{-1}(F)$. Thus $h^{-1}(F)$ is convex. Given a convex combination $u = \alpha_1 u_1 + \alpha_2 u_2$ with $u \in h^{-1}(F)$ and $u_1, u_2 \in \bar{co}(U_{p,q})$, some convex combination of $h(u_1), h(u_2)$ equals the point $h(u)$ in the face F, whence each $h(u_i) \in F$, and so each $u_i \in h^{-1}(F)$. Thus $h^{-1}(F)$ is a face of $\bar{co}(U_{p,q})$. Therefore h^{-1} preserves faces. By symmetry, h preserves faces as well. As extreme points correspond to singleton faces, h and h^{-1} also preserve extreme points. $\square$

Proof of Proposition 2. Let $\succsim$ be a binary relation on $\mathcal{P}(X)$ that satisfies axioms A1-A6. $\succsim$ satisfies axioms A1-A5, so by Theorem 1, there exists an expected Richter-Peleg multi-utility representation by a non-empty compact set $U \subseteq C(X)$. We need to show that $\bar{co}(U_{p,q})$ is a Choquet simplex and that the set $\partial_e \bar{co}(U_{p,q})$ of its extreme point is closed. Set $\mathcal{C} = \{\lambda(p-q) \mid \lambda \geq 0,\ p \succeq q\}$ and let $\mathcal{S}$ stand for the linear span of $\mathcal{P}(X) - \mathcal{P}(X)$. It is routine to show that $\mathcal{C}$ defines a preorder on $\mathcal{S}$. Equipped with such a preorder, denoted $\succsim$ by a slight abuse of notation, $\mathcal{S}$ becomes a preordered vector space and $\mathcal{S}/\sim$ an ordered vector space. $U \subseteq C(X)$ defines a multi-utility representation on $\mathcal{S}$ and a multi-utility representation $\tilde{U}$ on its quotient space.

Claim 1. *$\forall x \in S$, $\exists y \in S$ such that $y \succsim x, 0$ and if $z \in S$ is such that $z \succsim x, 0$, then $z \succsim y$.*

Proof. Let $x \in S$, there exist $\lambda \geq 0$ and $p, q \in \mathcal{P}(X)$ such that $x = \lambda(p - q)$. By axiom A6, there exist $r_1, s_1 \in \mathcal{P}(X)$ and $\alpha \in (0,1)$ such that $r_1 \succsim s_1$, and $\alpha r_1 + (1-\alpha)q \succsim \alpha s_1 + (1-\alpha)p$. Set $y = \frac{\lambda\alpha}{1-\alpha}(r_1 - s_1)$. It is clear that $y \succsim x, 0$. By axiom A6, again, if $r_2, s_2 \in \mathcal{P}(X)$ and $\beta \in (0,1)$ are such that $r_2 \succsim s_2$, and $\beta r_1 + (1-\beta)q \succsim \beta s_1 + (1-\beta)p$, then

$$\frac{\beta(1-\alpha)}{\alpha - 2\alpha\beta + \beta} r_2 + \frac{\alpha(1-\beta)}{\alpha - 2\alpha\beta + \beta} s_1 \succsim \frac{\beta(1-\alpha)}{\alpha - 2\alpha\beta + \beta} s_2 + \frac{\alpha(1-\beta)}{\alpha - 2\alpha\beta + \beta} r_1.$$

That is, if $z \in S$ is such that $z \succsim x, 0$, then $z \succsim y$. $\square$

Let us consider the vector space $\mathcal{L}(\mathcal{S}/\sim, \mathbb{R})$ of all homomorphisms from $\mathcal{S}/\sim$ to $\mathbb{R}$. Define $\succsim^*$ on $\mathcal{L}(\mathcal{S}/\sim, \mathbb{R})$ as follows: if $f, g \in \mathcal{L}(\mathcal{S}/\sim, \mathbb{R})$, then $f \succsim^* g$ iff $f(x) \geq g(x)$ for all $x \in (\mathcal{S}/\sim)^+$, where $(\mathcal{S}/\sim)^+$ is the positive cone of $\mathcal{S}/\sim$. It is trivial to see that $\succsim^*$ is a preorder. By claim 1, $\mathcal{S}/\sim$ is a lattice ordered vector space. Therefore, $\mathcal{S}/\sim$ is directed. It follows that $\succsim^*$ is a partial order.

Let $\mathcal{L}(\mathcal{S}/\sim, \mathbb{R})^+$ be the positive cone of $\mathcal{L}(\mathcal{S}/\sim, \mathbb{R})$ and let

$$B = \left\{ f \in \mathcal{L}(\mathcal{S}/\sim, \mathbb{R}) \;\middle|\; f = g - h; g, h \in \mathcal{L}(\mathcal{S}/\sim, \mathbb{R})^+ \right\}$$

denote the set of all relatively bounded homomorphisms from $\mathcal{S}/\sim$ to $\mathbb{R}$. As $\mathcal{S}/\sim$ is a lattice ordered vector space, by the Riesz-Kantorovich Theorem (see Aliprantis and Tourky (2007, Theorem 1.59)), B is a lattice ordered vector space. Note that B^+ is the set of all order preserving homomorphisms from $\mathcal{S}/\sim$ to $\mathbb{R}$. There is no difficulty to show that $\bar{co}(\tilde{U})$ is a base for B^+. Summing up, $\bar{co}(\tilde{U})$ is a base for a lattice cone in some real linear space, hence $\bar{co}(\tilde{U})$ is a Choquet simplex. It follows that $\bar{co}(U_{p,q})$ is a Choquet simplex.

It remains only to show that the set $\partial_e \bar{co}(U_{p,q})$ of its extreme point is closed.

Claim 2. *$\exists e \in S$, $\forall x \in S$, $\exists \alpha > 0$, $\alpha e \succsim x$.*

Proof. For a normalized representation $U_{p,q}$ with $p \succ q$, set $e = p - q$. By axiom A.4 (Open-continuity), there exist open neighbourhood N_p, N_q of p, q (respectively) such that $r \succ s$ for all $(r,s) \in N_p \times N_q$. Therefore, for all $l, m \in \mathcal{P}(X)$ there exists $\lambda \in (0,1)$ such that $\lambda p + (1-\lambda)m \succsim \lambda q + (1-\lambda)l$. If $x = \mu(l - m)$ with $\mu \geq 0$, put $\alpha = \frac{\lambda\mu}{1-\lambda}$. It is easy to verify that $\alpha e - x \in \mathcal{C}$, that is, $\alpha e \succsim x$. $\square$

By claims 1 and 2, $\mathcal{S}/\sim$ is a lattice ordered vector space with order unit. By the Stone-Kakutani-Krein-Yosida Theorem (see Alfsen (2012, Theorem II.1.10)), $\partial_e \bar{co}(\tilde{U})$ is closed. It follows that $\partial_e \bar{co}(U_{p,q})$ is closed, proving that $\bar{co}(U_{p,q})$ is a Bauer simplex. $\square$

Proof of Proposition 3. Let $\succsim$, $\succsim^o$ be two binary relations on $\mathcal{P}(X)$. Suppose that $\succsim$ satisfies A1-A6, $\succsim^o$ satisfies A0-A4, and that the pair $(\succsim, \succsim^o)$ jointly satisfy A7. Let $p^*, q^* \in \mathcal{P}(X)$ be such that $p^* \succ q^*$. By proposition 2, there exists an expected Richter-Peleg multi-utility representation for $\succsim$ by a Bauer simplex $B_{p^*,q^*} \subseteq C(X)$, that is, for all $p, q \in \mathcal{P}(X)$

(i) $p \succsim q \Leftrightarrow \mathbb{E}(p,u) \geq \mathbb{E}(q,u), \;\; \forall u \in B_{p^*,q^*}$.

Define $U_{p^*,q^*} = \partial_e B_{p^*,q^*}$. By proposition 1, U_{p^*,q^*} is a an expected Richter-Peleg multi-utility representation for $\succsim$ in the above sense. By the expected utility theorem, there exists an unique real-valued function $v \in C(X)$ that satisfies $\mathbb{E}(p^*, v) = 1$ and $\mathbb{E}(q^*, v) = 0$ such that for all $p, q \in \mathcal{P}(X)$

(ii) $p \succsim^o q \Leftrightarrow \mathbb{E}(p,v) \geq \mathbb{E}(q,v)$.

By axiom A7, we observe that $v \in B_{p^*,q^*}$. Therefore we can apply the Bauer Theorem (see Alfsen (2012, Theorem II.4.11)), there is a unique regular, positive and normalized Borel probability measure μ supported by U_{p^*,q^*} such that $L(v) = \int_{U_{p^*,q^*}} L(u)\mathrm{d}\mu(u)$ for every continuous, linear functional L on $C(X)$. Applying the usual representation theorem, it is straightforward to see that

(iii) $\mathbb{E}(p,v) = \displaystyle\int_{U_{p^*,q^*}} \mathbb{E}(p,u)\mathrm{d}\mu(u)$.

Observe that, by proposition 1, U_{p^*,q^*} unique up to homeomorphism. $\square$

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