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ERROR IN MEASUREMENT THEORY

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Abstract

In the standard theory of extensive measurement, a set of assumptions, or qualitative axioms, are formulated in terms of an ordering and a concatenation operation that lead to a deterministic additive scale unique up to a positive similarity transformation. In this paper, we extend the qualitative primitives of the theory of extensive measurement in such a way that the objects of the domain are represented by random variables and their expectations.

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1 Introduction

In the standard theory of extensive measurement, a set of assumptions, or qualitative axioms, are formulated in terms of an ordering $\succsim$ (of objects with respect to some property), generally a weak order, that is, it is transitive and any two objects are comparable, and a concatenation operation $\circ$ (between objects) that lead to a scale ψ unique up to a positive similarity transformation and satisfying

(i) $a \succsim b \Leftrightarrow \psi(a) \geq \psi(b)$,

(ii) $\psi(a \circ b) = \psi(a) + \psi(b)$.

The central idea of the theory of random quantities is to replace the additive scale by a random variable. This means that each object is assigned a random variable. In the case of extensive quantities, the expectation of the random variable replaces the usual additive scale, and the distribution of the random variable reflects the variability of the property in question, which could be intrinsic to the object or environment, or due to instrumental errors of observation.

The standard axiomatic theory of extensive measurement permits the construction of a numerical assignment which has a perfect accuracy. That is, the numerical assignment is the errorless value with respect to an unit and, when repeated ad infinitum, it is invariable. In the practice of extensive measurement, it is assumed that a series of replicas of any given object can be found.

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Once a unit is fixed, for an observation, if k replicas of the unit are equivalent to n replicas of an object a, then the measure of the object a is $\frac{k}{n}$.[1] Repeated observations may not yield the same results, for instance a second observation may yield to the measure $\frac{k+1}{n}$. Hence for the first observation, n replicas of a are strictly less than $k+1$ replicas of the unit while for the second observation n replicas of a are strictly largest than k replicas of the unit. Clearly, comparability is a very strong idealization as errors imply a form of incomparability between the objects. It becomes natural to use a quasi-order (of objects with respect to some property) $\succsim_s$ instead of a weak order to reflects comparisons, which are not subject to errors, of an observable attribute, that is, a transitive and reflexive binary relation. Quasi-orders[2] are represented by real-valued vectors with the pointwise ordering or which is equivalent by means of a set Φ of additive scale satisfying

(i) $a \succsim_s b \Leftrightarrow \phi(a) \geq \phi(b), \;\; \forall \phi \in \Phi$,

(ii) $\forall \phi \in \Phi$, $\phi(a \circ b) = \phi(a) + \phi(b)$.

Once the unit is fixed (that is, $\forall \phi \in \Phi$, $\phi(u) = 1$ for some u), we can assign a set of values for the possible measure of a measured object. In other words, there is a set of deterministic measurements which represents the measurement without errors. It is expected that this set of deterministic measurements is a sample space, under certain conditions which will be discussed later, for the measurement process. That is, the quasi-order encodes the result of all possible certain observations. At this point, we have almost a random variable because Φ is not yet a probability space. To do so, we introduce a supplementary qualitative ordering $\succsim_m$ on the set of objects which reflects average comparisons of an observable attribute with possible errors. Put differently, as an example, suppose that you observe in N trials if the attribute of an object a is greater or equal than the attribute of an object b, and that the results are variable. If in M trials the attribute of a is greater or equal than the attribute of b and in $N-M$ trials it is the opposite, then we can make a comparison in terms of relative-frequency. It is natural to ask if $\succsim_m$ admits a representation which is related, in a probabilistic way, to that of $\succsim_s$. To make things a bit more precise, suppose that $\succsim_s$ admits a representation by a set Φ of additive scale as above and let us denote by A the set of objects and by $\tilde{a}$ the real-valued function on Φ defined by $\tilde{a}(\phi) = \phi(a)$ for all $a \in A$. The representation notion we suggest requires a probability measure μ on some algebra of sets on Φ satisfying

$$a \succsim_m b \Leftrightarrow \int_\Phi \tilde{a}(\phi) \mathrm{d}\mu(\phi) \geq \int_\Phi \tilde{b}(\phi) \mathrm{d}\mu(\phi).$$

That is, each object is assigned a random variable and the expectation of the random variable replaces the usual additive scale.

Works that combines the qualitative structural analysis of measurement procedures and the analysis of error has been a neglected area of representational measurement theory even though its importance is generally acknowledged. For

[1] Let u be an unit, if $ku \sim na$, then $\psi(a) = \frac{k}{n}\psi(u) = \frac{k}{n}$.

[2] It is well-known (See Dushnik and Miller (1941) or Carlson (2008) for the theory of extensive measurement) that a quasi-order is the intersection of all its weak-ordered extensions.

instance, Luce and Narens (1994) devote several of their fifteen problems concerning the foundations-of-measurement to this subject. Some attempts are the approach of Falmagne (1980) of probabilizing axioms of measurement that gave rise to a series of papers by Falmagne and his collaborators on random variable representations for extensive and related measurement (cf., for instance, Iverson and Falmagne (1985)). The approach of Suppes and Zanotti (1992) in attempting to axiomatize qualitative moment information and to use that to characterize a random variable representation in extensive measurement. A survey of these two works can be found in Chapter sixteen of the second volume of Foundations of measurement (Suppes et al. (1978)). Other approaches connected with the probabilistic version of extensive measurement or with specific distributional results can be found in Heyer and Niederée (1989), Niederée and Heyer (1997), Regenwetter and Marley (2001) or Suck (1989, 1998, 2001), respectively.

We begin in the next section by discussing the axioms we shall study. Then, in section 3, we show that the axioms allow that the objects are represented by random variables. We conclude in section 4 in discussing some relaxations of our axioms. Finally, we go on in section 5 to present the proofs.

2 Axioms

Throughout this paper, we deal with five primitives: a nonempty set A, two binary relations $\succsim_s$ and $\succsim_m$ on A, a closed binary operation $\circ$ that maps $A \times A$ into A, and u a distinguished element of A. The interpretation is: A is a set of objects that exhibit the attribute observed; $a \succsim_s b$ holds if and only if a exhibits, *surely*, at least as much of the attributes as b; $a \succsim_m b$ holds if and only if a exhibits, *on average*, at least as much of the attributes as b; $a \circ b$ is an object in A that is obtained by concatenating a and b in some directed, ordered fashion; and u is an object that exhibits surely the attribute observed and taken as the unit.

Two related points need to be emphasized. First, the above defined relations are neither empty nor trivial. For instance, suppose that we are judging qualitative weight by deflections of an equal-arm pan balance. When a weight, heavier than the edge of sensitivity of the balance, is placed in one of the two pans, then it is quite likely that we will observe a deflection of the side where the object has been placed. Moreover it is extremely improbable that this observation fluctuates. Consequently, surely and in mean a material weight is heavier than the void weight. Second, these primitives are, as in the standard extensive measurement theory, idealization of the reality and do not tangibly exist. To state the previous point, it was implicitly assumed that the weight does not vary intrinsically and that the balance adjustment is stable over time. A consistent apparatus which describes our primitives in the example of weight measurement could be: the elements of A are invariable material objects; $a \succsim_s b$ is established by placing, repeatedly, a and b on the two pans of an equal-arm pan balance and observing, ad infinitum, which pan systematically drops; $a \succsim_m b$ is established by placing, repeatedly, a and b on the two pans of the same equal-arm pan balance and observing, ad infinitum, which pan drops on average; $a \circ b$

means that a and b are both placed in the same pan with a beneath b; and u a non-zero material object. We may doubt of our capacity to perform endless observation, provided that their results exist. However, in ordinary practice, the existence of expensive and laborious measurement techniques gives meaning to our definitions.

As usual, if $\succsim$ is a binary relation, we write $a \sim b$ if and only if $a \succsim b$ and $b \succsim a$; and $a \succ b$ if and only if $a \succsim b$ and not $(b \succsim a)$. For all $a \in A$ and n positive integer, na is defined inductively as: $1a = a$, and $(n+1)a = na \circ a$. We begin by listing and, when necessary, discussing the several axioms we shall study for the binary relation $\succsim_s$.

Definition 1. *Let A be a nonempty set, $\succsim_s$ a binary relation on A, $\circ$ a closed binary operation on A. The triple $\langle A, \succsim_s, \circ \rangle$ is a quasi-ordered Riesz closed extensive structure iff the following six axioms are satisfied for all $a, b, c, d, x, y, z, t \in A$:*

1. *Quasi-order: $\langle A, \succsim_s \rangle$ is a quasi-order, i.e., $\succsim_s$ is a transitive and reflexive relation.*
2. *Weak associativity and weak commutativity: $a \circ (b \circ c) \sim_s (a \circ b) \circ c$ and $a \circ b \sim_s b \circ a$.*
3. *Monotonicity: $a \succsim_s b$ iff $a \circ c \succsim_s b \circ c$.*
4. *Riesz interpolation property: If $a \circ y \succsim_s b \circ x$, $a \circ t \succsim_s b \circ z$, $c \circ y \succsim_s d \circ x$ and $c \circ t \succsim_s d \circ z$, then there exist $f, g \in A$ such that $a \circ g \succsim_s b \circ f$, $c \circ g \succsim_s d \circ f$, $f \circ y \succsim_s g \circ x$ and $f \circ t \succsim_s g \circ z$.*
5. *Archimedean: If $a \succ_s b$, then for any $c, d \in A$, there exists a positive integer n such that $na \circ d \succsim_s nb \circ c$.*
6. *Strongly Archimedean: If for any positive integers $na \circ d \succsim_s nb \circ c$, then $a \succsim_s b$.*

The first four axioms are quite standard in algebra and define what may be called a partially ordered Riesz semigroup. It is worth to say that Axiom 4 is the ordinary Riesz interpolation property for differences. For if we define $ab \succeq cd$ to mean $a \circ d \succsim_s b \circ c$, then Axiom 4 simply says that if $ab, cd \succeq xy, zt$, then there exists fg such that $ab, cd \succeq fg \succeq xy, zt$. It is a richness assumption on $\succsim_s$ and not on the set A as this axiom is trivially satisfied by a connected binary relation. It is difficult to say just what this means other than to restate it in words. If the differences ab and cd both exhibits, surely, at least as much of the attributes as xy and zt, then the axiom asserts that there is a difference fg that is surely between them. Axiom 5 is the usual Archimedean property which is equivalent to Axiom 6 if $\succsim_s$ is a connected binary relation. If $\succsim_s$ is actually not connected, then Axiom 5 states that the difference between two elements is surely bounded by the sum of a sufficient number of a sure positive difference replicas. By choosing a unit u which satisfies $u \circ a \succ_s a$ for all $a \in A$ (To be sure that u is a non-zero material object), this axiom implies that each element is surely bounded by some multiple (possibly very large) of u. Thereby, if a deterministic measurement can not be performed, deterministic bound exists for this measurement. In the case of weight measurement, when objects are placed in the two pans and do not cause a deflection of the arm from the horizontal, it is always possible to add a sufficient number of replicas of non-zero material object in one of the arms to observe, surely, a deflection. Axiom 6 has the role of the usual Archimedean property.

We turn now to the axioms for the binary relation $\succsim_m$. The axioms are those of a closed extensive structure. Therefore, the following definition is identical to the definition 3.1 of Krantz et al. (1976, p.73).

Definition 2. *Let A be a nonempty set, $\succsim_m$ a binary relation on A, $\circ$ a closed binary operation on A. The triple $\langle A, \succsim_m, \circ\rangle$ is a closed extensive structure iff the following four axioms are satisfied for all $a, b, c, d \in A$:*

1. *Weak order: $\langle A, \succsim_m\rangle$ is a weak order, i.e., $\succsim_s$ is a reflexive, transitive and connected relation.*
2. *Weak associativity: $a \circ (b \circ c) \sim_m (a \circ b) \circ c$.*
3. *Monotonicity: $a \succsim_m b$ iff $a \circ c \succsim_m b \circ c$ iff $c \circ a \succsim_m c \circ b$.*
4. *Archimedean: If for any positive integers $na \circ d \succsim_m nb \circ c$, then $a \succsim_m b$.*

Note that as $succsim_m$ is supposed to be a weak order, then we suppose explicitly that average comparisons can be done.

Our goal is to provide qualitative axioms such that we can prove that object a can be represented by a random variable and that its expectation replaces the additive scale. To do this, we will suppose three supplementary axioms. First, we suppose that A is countable. We make this assumption for mathematical convenience to avoid rather long technical developments. Despite the apparent restrictiveness of this assumption (i.e., despite the simplicity of the class of structures captured by our theory), we emphasize its realistic appearance in laboratory although the structure is supposed closed. For instance, two distinct elements a and b generate an infinite (countable) free semigroup, and this is still true for a countable set of distinct elements $\{a_1, a_2, \ldots\}$. Second, we postulate that there exists surely a non-zero element which will be the unit. Third, we assume a consistency requirement asserting that if a exhibits, surely, at least as much of the attributes as b, then a exhibits, on average, at least as much of the attributes as b. That is, the average observation does not question the sure observation. We have now discussed all the axioms, and we summarize them as follows:

Definition 3. *Let A be a nonempty set, $\succsim_s$ and $\succsim_m$ binary relations on A, $\circ$ a closed binary operation on A, and $u \in A$. The structure $\langle A, \succsim_s, \succsim_m, \circ, u\rangle$ is a non-deterministic closed extensive structure iff the following five axioms are satisfied for all $a, b \in A$:*

I. *A is countable.*

II. *The structure $\langle A, \succsim_s, \circ\rangle$ is a quasi-ordered Riesz closed extensive structure.*

III. *The structure $\langle A, \succsim_m, \circ\rangle$ is a closed extensive structure.*

IV. *$u \circ a \succ_s a$.*

V. *If $a \succ_s b$, then $a \succ_m b$; and if $a \sim_s b$, then $a \sim_m b$.*

3 Statement of the principal result

In this section, we state and discuss the significance of the principal results, leaving proofs for section 5.

To formulate the representation, it is convenient to introduce the following notion:

Definition 4. *Suppose that $\langle A, \succsim, \circ \rangle$ is a quasi-ordered Riesz closed extensive structure and that $u \in A$ satisfies $u \circ a \succ a$. A set Φ of real-valued functions on A is called a u-normalised multi-representation iff, for all $a, b \in A$,*

(i) $a \succsim b \Leftrightarrow \phi(a) \geq \phi(b), \ \ \forall \phi \in \Phi$;

(ii) $\forall \phi \in \Phi$, $\phi(a \circ b) = \phi(a) + \phi(b)$;

and

(iii) $\forall \phi \in \Phi$, $\phi(u) = 1$.

It is convenient to see the set Φ as a subset of the real vector space $\mathbb{R}^A$ of all real-valued functions on A. In addition, we equip $\mathbb{R}^A$ with the product topology, so that it becomes a locally convex Hausdorff space. In the reminder, we will sacrifice some straightforward formality for brevity and, all topological notions refers directly to this topology or to the relative topology induced by it. Moreover, if $\succsim$ admits a u-normalised multi-representation, then we denote by $\tilde{a}$, for all $a \in A$, the real-valued function on Φ defined by $\tilde{a}(\phi) = \phi(a)$.

Note that a closed extensive structure is a quasi-ordered Riesz closed extensive structure. It follows that, in this case, a u-normalised multi-representation is a singleton. We are now ready to state the representation that we want to prove.

Proposition 1. *Let A be a nonempty set, $\succsim_s$ and $\succsim_m$ binary relations on A, $\circ$ a closed binary operation on A, and $u \in A$. Suppose that $\langle A, \succsim_s, \succsim_m, \circ, u \rangle$ is a non-deterministic closed extensive structure. Then there exist an unique set Φ of real-valued functions on A and an unique probability measure μ on $\mathcal{B}(\Phi)$ (the algebra of Borel sets of Φ) such that Φ is a u-normalised multi-representation which represents $\succsim_s$ and that for all $a, b \in A$*

$$a \succsim_m b \Leftrightarrow \int_\Phi \tilde{a}(\phi) \mathrm{d}\mu(\phi) \geq \int_\Phi \tilde{b}(\phi) \mathrm{d}\mu(\phi).$$

Because the construction of the set Φ and the probability measure μ embodies the basic processes of extensive measurement, we first give a brief outline of the proof and then we will discuss the significance of the proposition. We naturally, by embedding, reduce the study of $\langle A, \succsim_s, \circ \rangle$ to the study of a partially ordered group G. The positive unit that was selected generates a totally ordered subgroup. Each element of the group is bounded above and below by some multiple of the unity, hence we can provide for each element an interval of possible real values (with respect to the chosen unit). By a non-constructive process, it is easy to construct one-way representations for this group, that is, a set Φ_o of additive real-valued functions which satisfies:

$$x \succeq y \Rightarrow \phi(x) \geq \phi(y), \ \forall \phi \in \Phi_o$$

It turns out that the inverse statement is true and that there exists Φ_o, a u-normalised multi-representation which represents $\succsim_s$. Φ_o is a subset of the real vector space $\mathbb{R}^G$ of all real-valued functions on G. We equip $\mathbb{R}^G$ with the product topology, so that it becomes a locally convex Hausdorff space. We then show that the closure of the convex hull of Φ_o is the largest set of additive real-valued functions which represents $\succsim_s$. As the closure of the convex hull of Φ_o is a compact convex subset of $\mathbb{R}^G$, we can apply the Krein-Milman theorem, that is, the closure of the convex hull of Φ_o is the closed convex hull of its extreme points. It follows that there exists a tiniest set Φ which represents $\succsim_s$. We have thus built, for each element a, a function $\tilde{a}$ from Φ (the sample space) to $\mathbb{R}$. In order that $\tilde{a}$ becomes a random variable, it remains to build a probability space on Φ. It is easy to see that there exists ψ which represents $\succsim_m$. By axiom V, ψ belongs to the closure of the convex hull of Φ. We prove next that the closure of the convex hull of Φ is a metrizable compact convex subset of $\mathbb{R}^G$. It follows that the Choquet theorem applies and that there is a probability measure μ on the closure of the convex hull of Φ which represents ψ and such that $\mu(\Phi) = 1$. To show the uniqueness of μ, we observe that the closure of the convex hull of Φ is a Choquet simplex. Therefore, for each element a, $\tilde{a}$ is now a measurable function from Φ to $\mathbb{R}$, i.e., a random variable.

The significance of this proposition is that errorless observations set a sample space for the measurement and the possible values that can be assigned to the measured objects. Moreover, each assignment of consolidated values for the measured objects carries a probability measure which reflects distribution of the measurement error. That is, past observation justify a probabilistic model of measurement. The main apparent drawback of our approach is that the distribution of the random variable associated to the element $a \circ b$ is not the convolution of the distributions of those associated to a and to b, i.e.,

$$\widetilde{a \circ b} \neq \tilde{a} + \tilde{b}$$

where $+$ represents the random variable summation operator, although they have an equal expectation, i.e.,

$$\psi(a \circ b) = \psi(a) + \psi(b).$$

In fact, it is quite normal because we observe $a \circ b$ as an element of the set A and not as a composite element or a couple (a, b). In other words, our axiomatization allows the measure of the element $c = a \circ b \in A$, but not the composite measure of the elements a then b or b then a. Composite measurement can be carried out in the framework of additive conjoint measurement. Therefore, an extension and a positive result is possible.

We turn now to generalize the uniqueness assertion of the closed extensive structure theorem in our context. To do this we consider the effect of a change of unit in a non-deterministic closed extensive structure $\langle A, \succsim_s, \succsim_m, \circ, u \rangle$.

Proposition 2. *Let A be a nonempty set, $\succsim_s$ and $\succsim_m$ binary relations on A, $\circ$ a closed binary operation on A, and $u, v \in A$. Suppose that $\langle A, \succsim_s, \succsim_m, \circ, u \rangle$ and*

$\langle A, \succsim_s, \succsim_m, \circ, v\rangle$ are non-deterministic closed extensive structures represented, in the above sense, by (Φ, μ) and (Φ', ν) respectively. If there exist positive integers k, n such that $nu \sim_s kv$, then there exists an homeomorphism $h : \Phi \to \Phi'$ such that $r(\phi) = \frac{k}{n}\phi$ for all $\phi \in \Phi$ and $\mu = \nu$.

The significance of the uniqueness assertion is that we may change the unit of measurement provided that they are surely commensurable. Another way of saying this is that the set of unit generates a closed extensive structure, that is, generate a subset of A for which $\succsim_s$ is connected. Without this assumption, it turns out that there still exist an (non-affine) homeomorphism between Φ and Φ', in other words, the sample space is an invariant but we need to use a Radon-Nikodym type theorem to go from one representation to another. This is logical because we must express a new probability measure as an integral with respect to a reference measure.

4 Concluding remarks

We now discuss the most relevant relaxations of our axioms. The two most restrictive axioms are in the Definition 1, namely the Riesz interpolation property and the Archimedean property (Definition 1, Axioms 4 and 5).

We begin by the Archimedean axiom named like that because it is identical to the Archimedean property of the standard extensive measurement theory. This axiom states that the difference between two elements is surely bounded by the sum of a sufficient number of a sure positive difference replicas. This axiom has two fundamental consequences. First, once we chose a unit (surely positive element), this axiom implies that any $a \in A$ satisfies $nu \succsim_s a$ and $nu \circ a \circ c \succsim_s c$ for some $c \in A$ and some positive integer n, whence $|\phi(a)| \leq n$ for all $\phi \in \Phi$. Hence, for any sequence $\alpha_1, \alpha_2, \ldots$ of non-negative real numbers such that $\sum \alpha_i = 1$, the series $\sum \alpha_i \phi_i(a)$ converges absolutely, for any Φ-valued sequence $\phi_1, \phi_2, \ldots$. More intuitively, the possible values that can be assigned to a measured object are not too scattered and their expectations (with respect to a countable probability space) exists. Without this assumption, the possible values that can be assigned to a measured object can be unbounded. In this case, it would be necessary to transform the axiom to have sufficiently concentrated values. The second consequence of this axiom, more theoretic, is that for any surely positive element the former analysis remains true. Therefore any surely positive element can be chosen as the unit of the measurement.

We continue with the Riesz interpolation property. If we relax this assumption, then there are two problems. First, the probability measure μ is no longer supported by the set Φ, there exist larger sets which includes Φ which support the measure μ. Second, the probability measure μ is not unique. Therefore, we can say that there exists some uncertainty about the sample space and about the distribution of the measurement error.

Finally, the present development may be compared with previous theories of extensive measurement. The advantage over "standard" theories, which assume comparability, is fairly obvious: those theories do not give a complete account of the procedures required to deal with variability of the comparison process. By contrast, the present work shows how to infer the random variable representation without much departure from the standard theory. Suppes and Zanotti (1992) also considered the problem of inexact measurement. He gives qualitative primitives for moments of the objects. In the present approach, we give qualitative axioms for the set of objects.

5 Proofs

5.1 Preliminary lemmas

To prove Propositions 1 and 2, we shall need several lemmas. The hypothesis common to all the lemmas is that $\langle A, \succsim_s, \succsim_m, \circ, u\rangle$ is a non-deterministic closed extensive structure (Axioms I-V of Definition 3). Proofs of Lemmas 1 and 2 are omitted because they involve only routine use of the axioms.

By Axiom II, $\langle A, \succsim_s, \circ\rangle$ is a quasi-ordered Riesz closed extensive structure (Definition 1) and by Axiom IV, $u \in A$ satisfies $u \circ a \succ_s a$ for all $a \in A$. Define $\succsim_s^o$ on $A \times A$ as follows: for all $a, b, c, d \in A$,

$$(a, b) \succsim_s^o (c, d) \;\; iff \;\; a \circ d \succsim_s b \circ c.$$

Lemma 1. *The binary relation $\succsim_s^o$ on $A \times A$ is a quasi-order.*

Define:

$$[a, b] = \{(a', b') | a', b' \in A, \;\; (a', b') \sim_s^o (a, b)\},$$

$$G = A \times A / \sim_s^o,$$

$$* \;\; : [a, b] * [c, d] = [a \circ c, b \circ d],$$

$$[0] = [a, a], \; a \in A,$$

$$-[a, b] = [b, a],$$

$$[u^o] = [u \circ a, a], \; a \in A.$$

By abuse of notation, the induced order on G is denoted $\succsim_s^o$. Elements of G are denoted by $x, y, \ldots$, the zero element $[0]$ is denoted 0 and $[u^o]$ is denoted by u^o.

Lemma 2. *$\langle G, \succsim_s^o, *\rangle$ is a partially ordered commutative group which satisfies the following properties,*

- *Riesz interpolation: for all $x, y, z, w \in G$ such that $x, y \succsim_s^o z, w$, there exists $t \in G$ with $x, y \succsim_s^o t \succsim_s^o z, w$;*
- *Archimedean: for all $x, y \in G$ such that $x \succ_s^o 0$, there exists a positive integer n for which $nx \succsim_s^o y$;*
- *Strongly Archimedean: whenever $x, y \in G$ such that $nx \succsim_s^o y$ for all positive integers n, then $x \succsim_s^o 0$;*

- $u^o \succ_s^o 0$

Lemma 3. *Let Axioms II and IV hold.* $\langle G, \succsim_s^o, * \rangle$ *is unperforated, that is, if* $nx \succsim_s^o 0$ *for some positive integer, then* $x \succsim_s^o 0$.

Proof. Let $x \in G$ and $k \in \mathbb{N}$ such that $kx \succsim_s^o 0$. As G is Archimedean, for all $n = 1, \dots k-1$, there exists $m \in \mathbb{N}$ such that $mu^o \succsim_s^o n(-x)$, i.e., $nx \succsim_s^o m(-u)$.

Since u^o satisfies $u^o \succ_s^o 0$, then $nu^o \succ_s^o u^o \succ_s^o 0$ for all positive integers n, while $0 \succ_s^o -u^o \succ_s^o mu^o$ for all negative integers m. Hence, the subgroup generated by u^o is simply ordered.

Consequently, there exists an element $y \in G$ such that $nx \succsim_s^o y$ for each $n = 0, 1, \dots k-1$. Since $qkx \succsim_s^o 0$ for all positive integers q, it follows that $nx \succsim_s^o y$ for all positive integers n. As G is Strongly Archimedean, then $x \succsim_s^o 0$. Therefore G is unperforated. □

Lemma 4. *Let Axioms II and IV hold. Then there exists a set* Φ_o *of real-valued additive functions on* G *such that for all* $\phi \in \Phi_o$, $\phi(u^o) = 1$, *and for all* $x \in G$,

$$x \succsim_s^o 0 \Leftrightarrow \phi(x) \geq 0, \ \ \forall \phi \in \Phi_o.$$

Proof. Stage 1 (There exists ϕ a real-valued additive function on G such that $\phi(u^o) = 1$, and for all $x \in G$, $x \succsim_s^o 0 \Rightarrow \phi(x) \geq 0$). The subgroup generated by u^o, denoted $\langle u^o \rangle$, is simply ordered. Thus there exists ϕ_o a real-valued additive function on $\langle u^o \rangle$ such that $\phi_o(u^o) = 1$, and for all $x \in \langle u^o \rangle$, $x \succsim_s^o 0 \Rightarrow \phi_o(x) \geq 0$.

Let $\mathcal{D}$ be the collection of all pairs (H, ψ) such that H is a subgroup of G containing $\langle u^o \rangle$ and $\psi : H \to \mathbb{R}$ is a real-valued additive function on H extending ϕ_o, and such that, for all $x \in H$, $x \succsim_s^o 0 \Rightarrow \psi(x) \geq 0$. Clearly $\mathcal{D}$ is non-empty since it includes $(\langle u^o \rangle, \phi_o)$.

Define $\succsim$ on $\mathcal{D}$ as follows: if $(H, \psi), (H', \psi') \in \mathcal{D}$, then $(H', \psi') \succsim (H, \psi)$ if and only if $H \subseteq H'$ and ψ' extends ψ. It is trivial to see that $\succsim$ is a partial order for which any simply ordered nonempty subset of $\mathcal{D}$ has an upper bound in $\mathcal{D}$. Therefore, by Zorn's lemma, there exists a maximal element (K, ϕ), and we need only show that $K = G$.

If there exists an element $x \in G \setminus K$, set

$$p = sup \left\{ \frac{\phi(y)}{m} \;\middle|\; y \in K;\ m \in \mathbb{N};\ mx \succsim_s^o y \right\},$$
$$r = inf \left\{ \frac{\phi(z)}{n} \;\middle|\; z \in K;\ n \in \mathbb{N};\ z \succsim_s^o nx \right\}.$$

By Lemma 2 $\langle G, \succsim_s^o, * \rangle$ satisfies the Archimedean property, there exist some positive integers i, j such that $ju^o \succsim_s^o -x$ and $iu^o \succsim_s^o x$, that is, $x \succsim_s^o j(-u^o)$ and $iu^o \succsim_s^o x$. As $iu^o, j(-u^o) \in K$, it follows that $p \geq -j$ and $i \geq r$. Moreover, if there exist $y, z \in K$ and $m, n \in \mathbb{N}$ such that $mx \succsim_s^o y$ and $z \succsim_s^o nx$, then $mz \succsim_s^o nmx \succsim_s^o ny$ and so $m\phi(z) \geq n\phi(y)$, whence $r \geq p$. Thus $i \geq r \geq p \geq -j$. We show that for a real number $q \in [p, r]$, ϕ can be extended to a real-valued additive function $\phi' : K + \mathbb{Z}x \to \mathbb{R}$ such that $\phi'(x) = q$ and which satisfies for all $y \in K + \mathbb{Z}x$, $y \succsim_s^o 0 \Rightarrow \phi'(y) \geq 0$. We claim that if $v \in K$ and $k \in \mathbb{Z}$ such that $v + kx \succsim_s^o 0$, then $\phi(v) + kq \geq 0$. If $k = 0$, then $v \succsim_s^o 0$, and so $\phi(v) + kq = \phi(v) \geq 0$, by definition of ϕ. If $k > 0$, then as $kx \succsim_s^o -v$ we have $q \geq p \geq -\frac{\phi(v)}{k}$, whence $\phi(v) + kq \geq 0$. Similarly, if $k < 0$, then as $v \succsim_s^o (-k)x$ we have $-\frac{\phi(v)}{k} \geq r \geq q$, and again $\phi(v) + kq \geq 0$. Thus the claim is proved. Therefore ϕ extends to a well-defined real-valued additive function $\phi' : K + \mathbb{Z}x \to \mathbb{R}$. But then $(K + \mathbb{Z}x, \phi')$ is an element of $\mathcal{D}$ strictly above (K, ϕ), contradicting the maximality of (K, ϕ). Therefore $K = G$ and there exists ϕ a real-valued additive function on G such that $\phi(u^o) = 1$, and for all $x \in G$, $x \succsim_s^o 0 \Rightarrow \phi(x) \geq 0$, as desired. We denote by Φ_o the set of all real-valued additive functions on G such that for all $\phi \in \Phi_o$, $\phi(u^o) = 1$, and for all $x \in G$,

$$x \succsim_s^o 0 \Rightarrow \phi(x) \geq 0, \ \ \forall \phi \in \Phi_o.$$

We have shown that this set is non-empty.

Stage 2 (For all $x \in G$, $x \succsim_s^o 0 \Leftrightarrow \phi(x) \geq 0, \ \forall \phi \in \Phi_o$). By the previous stage, necessity is obvious, we prove sufficiency. Let $x \in G$ and assume that $\phi(x) \geq 0$ for all $\phi \in \Phi_o$. Set

$$p = sup \left\{ \frac{k}{m} \;\middle|\; k \in \mathbb{Z};\ m \in \mathbb{N};\ mx \succsim_s^o ku^o \right\}.$$

Recall that there exists ϕ_o a real-valued additive function on $\langle u^o \rangle$ such that $\phi_o(u^o) = 1$, and for all $x \in \langle u^o \rangle$, $x \succsim_s^o 0 \Rightarrow \phi_o(x) \geq 0$. We observe that

$$p = sup \left\{ \frac{\phi(y)}{m} \;\middle|\; y \in \langle u^o \rangle ;\ m \in \mathbb{N};\ mx \succsim_s^o y \right\}.$$

We can deduce, by the same reasoning as in the previous stage, that there exists $\psi \in \Phi_o$ such that $\psi(x) = p$ and hence $p \geq 0$. Given any $n \in \mathbb{N}$, there must exist $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that $mx \succsim_s^o ku^o$ and $\frac{k}{m} > -\frac{1}{n}$. Then $kn > -m$, whence $mnx \succsim_s^o knu^o \succ_s^o -mu^o$, and therefore $m(nx + u^o) \succ_s^o 0$. By Lemma 3, we have $nx + u^o \succ_s^o 0$ for all positive integers n, whence $x \succ_s^o 0$ because G is Strongly Archimedean, as desired. □

We view Φ_o as a subset of the real vector space $\mathbb{R}^G$ of all real-valued functions on G. We equip $\mathbb{R}^G$ with the product topology, so that it becomes a locally convex Hausdorff space. If S is a subset of $\mathbb{R}^G$, we denote by $cl(conv(S))$ the closure of the convex hull of S.

Lemma 5. *Let Axioms II and IV hold. If Φ_o is a representation of $\succsim_s^o$ as in Lemma 4, then $cl(conv(\Phi_o))$ is a compact convex subset of $\mathbb{R}^G$. Moreover, if Axiom I holds, then $cl(conv(\Phi_o))$ is metrizable.*

Proof. It is clear, by definition, that $cl(conv(\Phi_o))$ is closed convex subset of $\mathbb{R}^G$. For each $x \in G$, choose $n_x \in \mathbb{N}$ such that $n_x u^o \succsim_s^o x \succsim_s^o -n_x u^o$. Such n_x exists because G is Archimedean. For all $\phi \in cl(conv(\Phi_o))$, we have $n_x \geq \phi(x) \geq -n_x$, and hence

$$cl(conv(\Phi_o)) \subseteq \prod_{x \in G} [-n_x, n_x].$$

This product of compact intervals is compact by the Tychonoff Theorem. Therefore $cl(conv(\Phi_o))$ is a closed subset of a compact set, and consequently, it is compact.

By Axiom I, G is at most countable, whence $\mathbb{R}^G$ is metrizable. Hence, $cl(conv(\Phi_o))$ is metrizable. □

Lemma 6. *Let Axioms II and IV hold.* Φ_o *and* Φ'_o *are two representations of* $\succsim_s^o$ *as in Lemma 4 if and only if* $cl(conv(\Phi_o)) = cl(conv(\Phi'_o))$.

Proof. Since necessity is trivial, we shall prove here only the sufficiency. Suppose that there exists ϕ' such that $\phi' \in cl(conv(\Phi'_o)) \setminus cl(conv(\Phi_o))$. $\{\phi'\}$ and $cl(conv(\Phi_o))$ are disjoint nonempty subsets of $\mathbb{R}^G$. $\{\phi'\}$ is closed and $cl(conv(\Phi_o))$ is compact. we may apply the Hahn-Banach Theorem to find a $x \in G$ such that

$$sup\left\{f(x)|f = \phi'\right\} < 0 \leq inf\left\{f(x)|f \in \text{cl}(\text{Conv}(\Phi_o))\right\}.$$

Whence, we get $\phi'(x) < 0$ and $\phi(x) \geq 0$ for all $\phi \in \Phi_o$, which is a contradiction. Therefore, if Φ_o and Φ'_o represent $\succsim_s^o$ as in Lemma 4, then $cl(conv(\Phi_o)) = cl(conv(\Phi'_o))$. □

Definition 5. *A simplex in a linear space* V *is any convex subset* K *of* V *that is affinely isomorphic to a base for a lattice cone in some real linear space. A simplex* K *in a locally convex Hausdorff space is said to be Choquet if* K *is compact.*

Lemma 7. *Let Axioms II and IV hold. If* Φ_o *is a representation of* $\succsim_s^o$ *as in Lemma 4, then* $cl(conv(\Phi_o))$ *is a Choquet simplex.*

Proof. Since compactness has been already established (Lemma 5) and that $cl(conv(\Phi_o))$ is a convex subset of $\mathbb{R}^G$, we need to show that $cl(conv(\Phi_o))$ is affinely isomorphic to a base for a lattice cone in some real linear space.

Let us consider the group $\mathcal{H}(G, \mathbb{R})$ of all group homomorphisms from G to $\mathbb{R}$. Define $\succsim^*$ on $\mathcal{H}(G, \mathbb{R})$ as follows: if $f, g \in \mathcal{H}(G, \mathbb{R})$, then $f \succsim^* g$ iff $f(x) \geq g(x)$ for all $x \in G^+$, where G^+ is the positive cone of G. It is trivial to see that $\succsim^*$ is a quasi-order. For all $x, y \in G$, there exist n, m positive integers such that $nu^o \succsim_s^o x$ and $mu^o \succsim_s^o y$, whence $max\{nu^o, mu^o\} \succsim_s^o x, y$, hence G is directed. It follows that $\succsim^*$ is a partial order.

Let $\mathcal{H}(G, \mathbb{R})^+$ be the positive cone of $\mathcal{H}(G, \mathbb{R})$ and let

$$B = \left\{f \in \mathcal{H}(G, \mathbb{R}) \;\middle|\; f = g - h; g, h \in \mathcal{H}(G, \mathbb{R})^+\right\}$$

denote the set of all relatively bounded group homomorphisms from G to $\mathbb{R}$. G is directed and satisfies the Riesz interpolation property, it follows (See Riesz (1940, Theorem 1)) that B is a Dedekind complete lattice-ordered Abelian group. It is clear that B is also a partially ordered real vector space.

Note that B^+ is the set of all order preserving homomorphisms from G to $\mathbb{R}$. If $f = 0$, then $f = 0\phi$ for any $\phi \in cl(conv(\Phi_o))$. If $f \neq 0$, then $f(x) \neq 0$

for some $x \in G$. Choose n a positive integer such that $nu^o \succsim_s^o x \succsim_s^o -nu^o$, and observe that $nf(u^o) \geq f(x) \geq -nf(u^o)$, whence $f(u^o) > 0$. Define $\phi_f : G \to \mathbb{R}$ by $\phi_f(x) = (\frac{1}{f(u^o)})f(x)$, whence ϕ_f is a real-valued additive function on G such that $\phi_f(u^o) = 1$, and such that for all $x \in G$, $x \succsim_s^o 0 \Rightarrow \phi_f(x) \geq 0$. Hence $\phi_f \in cl(conv(\Phi_o))$, and $f(x) = f(u^o)\phi_f(x)$, that is,

$$B^+ = \{\alpha\phi \mid \alpha > 0 \; and \; \phi \in cl(conv(\Phi_o))\}.$$

$cl(conv(\Phi_o))$ lies in the hyperplane $\{f \in B \mid f(u^o) = 1\}$ which misses the origin. If $cl(conv(\Phi_o))$ is not a base for B^+, then some non-zero point $p \in$ may be written as $p = \alpha\phi = \beta\psi$ where α, β are distinct positive real numbers and ϕ, ψ are distinct points of $cl(conv(\Phi_o))$. Then the origin may be expressed as an affine combination of $\alpha\phi$ and $\beta\psi$. But then the origin belongs to the hyperplane $\{f \in B \mid f(u^o) = 1\}$, which is false. Therefore $cl(conv(\Phi_o))$ is a base for B^+. Summing up, $cl(conv(\Phi_o))$ is a base for a lattice cone in some real linear space, hence $cl(conv(\Phi_o))$ is a simplex. □

We are know ready to prove the propositions 1 and 2.

Proof of Proposition 1. We assume that Axioms I-V hold. Lemma 4 prove the existence of a set Φ_o of real-valued functions on G such that for all $[a, b], [c, d] \in G$

(i) $[a, b] \succsim_s^o [c, d] \Leftrightarrow \psi([a, b]) \geq \psi([c, d]), \;\; \forall\psi \in \Phi_o$,

(ii) $\forall\psi \in \Phi_o$, $\phi([a, b] * [c, d]) = \psi([a, b]) + \psi([c, d])$,

and

(iii) $\forall\psi \in \Phi_o$, $\psi([u \circ a, a]) = 1$.

By Lemma 5, $cl(conv(\Phi_o))$ is a compact convex subset of a locally convex Hausdorff space, the KreinMilman theorem assures that $cl(conv(\Phi_o))$ is the closed convex hull of its extreme points. Denote by Φ the set of all its extreme points, by Lemma 6 Φ is again a set of real-valued functions on G which satisfies (i), (ii) and (iii) and this set is unique. For each $\psi \in \Phi$, define ϕ on A as follows: for all $a \in A$, $\phi(a) = \psi([2a, a])$. It is straightforward to verify that we have constructed a set Φ which satisfies for all $a, b \in A$

(i) $a \succsim_s b \Leftrightarrow \phi(a) \geq \phi(b), \;\; \forall\phi \in \Phi$;

(ii) $\forall\phi \in \Phi$, $\phi(a \circ b) = \phi(a) + \phi(b)$;

and

(iii) $\forall\phi \in \Phi$, $\phi(u) = 1$.

By Axiom III and Theorem 3.1 of Krantz et al. (1976), there exists a real-valued function ϕ_m on A such that for all $a, b \in A$

(i) $a \succsim_m b \Leftrightarrow \phi_m(a) \geq \phi_m(b)$,

(ii) $\phi_m(a \circ b) = \phi_m(a) + \phi_m(b)$.

Define ψ_m on G as follows: for all $[a,b] \in G$, $\psi_m([a,b]) = \phi_m(a) - \phi_m(b)$. By Axiom V, it is easy to see that ψ_m is a real-valued additive function on G such that $\psi_m(u^o) = 1$, and that for all $[a,b],[c,d] \in G$, $[a,b] \succsim_s^o [c,d] \Rightarrow \psi_m([a,b]) \geq \psi_m([c,d]))$. That is, $\psi_m \in cl(conv(\Phi_o))$. By Lemma 5, $cl(conv(\Phi_o))$ is a metrizable compact convex subset of the locally convex Hausdorff space $\mathbb{R}^G$, so the Choquet theorem asserts that there exists a probability measure μ on $cl(conv(\Phi_o))$ which represents ψ_m and which is supported by the extreme points of $cl(conv(\Phi_o))$, i.e., $\mu(\Phi) = 1$. Moreover, according to Lemma 7, $cl(conv(\Phi_o))$ is a Choquet Simplex, hence μ is unique by the Choquet-Meyer. Thus, considering the sub-algebra generated by the Borel sets of Φ, it follows that for all $a \in A$

$$\phi_m(a) = \psi_m[2a,a] = \int_\Phi \psi[2a,a]\mathrm{d}\mu(\psi) = \int_\Phi \phi(a)\mathrm{d}\mu(\phi) = \int_\Phi \tilde{a}(\phi)\mathrm{d}\mu(\phi),$$

where $\tilde{a}$ is, for all $a \in A$, the real-valued function on Φ defined by $\tilde{a}(\phi) = \phi(a)$. □

Proof of Proposition 2. Suppose that $\langle A, \succsim_s, \succsim_m, \circ, u\rangle$ and $\langle A, \succsim_s, \succsim_m, \circ, v\rangle$ are non-deterministic closed extensive structures represented, as in the above Proposition, by (Φ, μ) and (Φ', ν) respectively, and that there exist positive integers k, n such that $nu \sim_s kv$. It is straightforward to show that $\phi(v) = \frac{n}{k}$ for all $\phi \in \Phi$, so there is no difficulty in constructing an homeomorphism $h : \Phi \to \Phi'$ such that $r(\phi) = \frac{k}{n}\phi$ for all $\phi \in \Phi$. According to the uniqueness of ϕ_m, we have $\mu = \nu$. □

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